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The axisymmetric equivalent of Kolmogorov's equation

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Abstract. A type of turbulence which is next to local isotropy in order of simplicity, but which corresponds more closely to turbulent flows encountered in practice, is locally axisymmetric turbulence. A representation of the second and third order structure function tensors of homogeneous axisymmetric turbulence is given. The dynamic equation relating the second and third order scalar structure functions is derived. When axisymmetry turns into isotropy, this equation is reduced to the well-known isotropic result: Kolmogorov's equation. The corresponding limiting form is also reduced to the well-known isotropic limiting form of Kolmogorov's equation. The new axisymmetric and theoretical results may have important consequences on several current ideas on the fine structure of turbulence, such as ideas developed by analysis based on the isotropic dissipation rate $\epsilon_{\rm iso}$ or such as extended self similarity (ESS) and the scaling laws for the n-order structure functions.

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1 Introduction

A fundamental result for the basic dynamic equations for the structure functions of isotropic turbulence is Kolmogorov's equation which connect the longitudinal scalar function of second-order, $\mathrm{D}_{LL}(\mathbf{r},t)=\langle(\delta u_L)^2\rangle$ to the third-order one, $\mathrm{D}_{LLL}(\mathbf{r},t)= \langle(\delta u_L)^3\rangle$, where $\delta u_L(r) = u_L(\mathbf{x}+\mathbf{r})-u_L(\mathbf{x})$. Kolmogorov [1] used Karman-Howarth's (K-H's) equation to derive his equation. It presupposes global isotropy (isotropy of the small scales of turbulence as well as of the large scales),

$$D_{LLL} = 6\nu\frac{\mathrm{d}D_{LL}}{\mathrm{d}r} - \frac{4}{5}\epsilon_{\rm iso}r \tag{1}$$

where ν is the kinematic viscosity, r the spacing or scale, $\langle\rangle$ the average over the pdf of $\delta u(r)$ and $\epsilon_{\rm iso}$ is the mean rate of energy dissipation. For the inertial range scales, (1) is reduced to the well-known Kolmogorov's $\frac{4}{5}$ law

$$D_{LLL} = -\frac{4}{5}\epsilon_{\rm iso}r. \tag{2}$$

This equation has received significant attention by experimentalists since it predicts a linear behaviour of $\langle(\delta u)^3\rangle$ in the inertial range.

For example, Antonia *et al.* [2] have measured the second-order structure functions in both laboratory and atmosphere. They indicate that the four-fifth law, equation (2), which is a consequence of the Navier-Stokes equations with the assumption of local isotropy, provides an important, although perhaps stringent, test for the data. Moreover, for all the flows, the isotropic dissipation, $\epsilon_{\rm iso} = 15\nu\overline{u_{1,1}^2}$, was assumed to determine the rate of dissipation ϵ. However, the authors pointed out that a more reliable estimate of ϵ can perhaps be obtained from the third-order structure function $\langle(\delta u)^3\rangle$ since the behavior of $\langle(\delta u)^3\rangle$ in the inertial sub-range can be obtained directly from the four-fifth law, equation (2).

Moisy *et al.* [3] have carried out a detailed systematic comparison between Kolmogorov equation with a forcing term and experimental measurements, in low temperature helium gas. They showed that a forced Kolmogorov equation is accurately verified by experiment, for a range of R_λ between 120 and 1200, within ±3% relative error. Their results, whether close or far from the asymptotics, have been accurately interpreted by assuming an isotropic homogeneous turbulence state.

In reference [4], the authors have modeled the velocity field in an isotropic and homogeneous turbulence. They presented velocity structure functions with scaling behaviors close to those known in the experiments and DNS. They remarked that Kolmogorov's four-fifth law is observed to be valid in a small scale range. Their model analysis is considered to represent successfully the statistical behaviors at small scales and higher orders.

The average turbulent energy dissipation is often estimated by assuming isotropy and measuring the temporal derivative of the longitudinal velocity fluctuation [5]. In this reference, the nine major terms that make up the total dissipation have been measured in the self-preserving

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region of a cylinder wake. The results indicate that local isotropy is not satisfied; the isotropic dissipation, computed by assuming isotropic relations, being smaller than the total dissipation by about 45% on the wake centerline and by 88% near the wake edge. When the assumption of isotropy is relaxed to one of local axisymmetry [6], satisfactory agreement is indicated by the data outside the wall-region. Prakovsky *et al.* [7] have tested experimentally the validity of Kolmogorov refined similarity hypothesis (RSH) in the mixing layer and in the return channel of a large wind tunnel. The energy dissipation rate is measured and the results are in good agreement with the RSH. Antonia *et al.* [8] showed that the fully measured mean dissipation rate ϵ is in good agreement with the value inferred from the rate of decay of the mean turbulent energy in the quasi-homogeneous region of a decaying grid turbulence. The isotropic mean energy dissipation rate $\epsilon_{\rm iso}$ agrees with this value even though individual elements of ϵ indicate departures from isotropy due to the small value of R_λ. Hosokawa *et al.* [9] revealed that the one dimensional (1D) surrogates of the scalar dissipation rate as well as the energy dissipation rate are often used in place of the true ones in investigations relevant to the Obukhov-Corrsin spectral form and the joint-multifractal nature of isotropic turbulence with an advected scalar. Their DNS show that the use of the 1D surrogates lead to fundamental changes in statistics.

In 1959, Monin [10] relaxed the assumption of isotropy to local isotropy. The notion of locally isotropic turbulence which is less restrictive than isotropy was introduced by Kolmogorv in 1941 [11]. However, it is important to emphasize that turbulent flows encountered in practice and that many laboratory flows are only approximately isotropic. The question of the degree of their isotropy is still not quite clear. Deviations from the isotropic state are more pronounced when the degree of anisotropy of a turbulent flow increases (real flow). In these special cases, an accurate description of real turbulent flow can be achieved by investigating more general theoretical models involving, for example, axisymmetric or general homogeneous but non-isotropic turbulence. That was the purpose of some authors [12–15]. To produce general results, proved on more general grounds than the previous isotropic results, these authors derived Monin's equation using only local homogeneity. Derivations were carried out using local homogeneity to the furthest extend possible. However, general locally homogeneous and non-isotropic turbulence is somewhat difficult to be tested experimentally. Indeed, in the case of homogeneous turbulence when no symmetry conditions at all are imposed, the statistics of a turbulent field are somewhat difficult to study [16,17]. Because of this, it is usual and practical as well as gain in simplicity, to consider fields of turbulence which satisfy certain symmetry conditions (in statistical sense).

Homogeneous axisymmetric turbulence is a type of turbulence which is next to local isotropy in order of simplicity, but which corresponds more closely to turbulent flows encountered in practice. It is the first test case for hypotheses on general effects of anisotropy. It would seem profitable to examine the form that Kolmogorov and Monin equations would take when only local axisymmetry is adopted, an assumption which is intermediate in severity between local homogeneity and local isotropy. Recently, Lindborg [18] gave an interesting theoretical representation of axisymmetric turbulence. He obtained expressions for second and third-order axisymmetric two-point correlation tensors in terms of measurable scalar functions. Following Lindborg's representation, Ould-Rouiss *et al.* [15] proposed preliminary investigations relative to the axisymmetric forms of the structure function tensors.

Moreover, the statistical behavior of 3D fully developed turbulence at small scales has been intensively investigated in the last five decades. A common way to study this problem is through the velocity structure functions $\langle(\delta u_L)^n\rangle$. Indeed, K41 theory predicts that at very large Reynolds numbers, the scaling $\langle(\delta u_L(r))^n\rangle \approx r^{\xi(n)}$ with $\xi(n) = n/3$ holds for r in the inertial range (*i.e.* $\eta \ll r \ll L$, η is the dissipative scale and L the integral scale). However, many experimental and numerical investigations have highlighted slight deviations from the K41 predictions $\xi(n) = n/3$, $n \neq 3$. These deviations are usually related to intermittency in the energy dissipation. Data of various and different turbulent flows at high Reynolds numbers have been analysed and showed the existence of universal scaling laws for $\langle(\delta u_L(r))^n\rangle$ in the inertial range (self-similarity). Recently, Benzi *et al.* [19] showed that the statistical properties are also self similar at low Reynolds numbers. An extended self scaling was found when a structure function is plotted against the other one [20]. Much work have been devoted in the last few decades to the measurement and modeling of the scaling of structure functions in turbulent flows. Qian [21] applied Kolomogorov equation to study the relative scaling of $D_{LL}(r)$ against $-D_{LLL}(r)$ according to the ESS method. The limits of ESS are discussed in reference [22]. A modification of the ESS concept is proposed. Corrections to the original ESS form are determined. Stolovitzky and Sreenivasan [23] indicated that the deviations from the Kolmogorov scaling appear real in the inertial range. Therefore, it would be interesting to examine the consequences of the new axisymmetric and theoretical results developed in the present paper on the extended-self-similarity theory.

This work is mainly motivated by experimental needs. The objective is to obtain expressions that can be tested experimentally. Thus, locally axisymmetric turbulence is the best alternative. It is an assumption intermediate between local homogeneity for which relevant quantities are not measurable and local isotropy which is the simplest case of turbulence or a mathematical idealization.

Following Lindborg's representation [18], this paper presents an analysis of homogeneous axisymmetric turbulence. The "axisymmetric" equivalent of the dynamic "isotropic" equation is derived. Axisymmetric forms of the second and third-order tensors for velocity structure functions are derived in Sections 3 and 4 respectively. The axisymmetric form of the mean energy dissipation rate is considered in Section 5. The "axisymmetric" equivalent of Monin's "isotropic" equation is derived in Section 6 as well as the limiting form of the axisymmetric form of Kolmogorov's equation when r → 0. The "axisymmetric" equivalent of Kolmogorov's "isotropic" equation is also derived in Section 6. Consequences on the fine structure of turbulence are discussed in Section 7.

2 Monin's equation

For homogeneous isotropic turbulence, Monin (see p. 403, Ref. [10]) has derived an equation relating $D_{ij} = \langle \delta u_i \delta u_j \rangle$, the second-order velocity increment tensor to $D_{ijk} = \langle \delta u_i \delta u_j \delta u_k \rangle$, the third-order velocity increment tensor. He assumed local isotropy to make the pressure terms vanish. After contracting indices ($i = j$), the equation was projected onto the longitudinal direction, leading to Kolmogorov's equation. Ould-Rouiss *et al.* [15] were able to derive Monin's equation (*i.e.* the non-projected form of Kolmogorov's equation) assuming only local homogeneity. The main elements of the derivation are briefly recalled below.

The difference between the Navier-Stokes equations for the velocity component u_{0i} at point x_{0i} and the component u_i at $x_i = x_{0i} + r_i$ is

$$\frac{\partial}{\partial t}\delta u_i + u_\alpha \partial_\alpha u_i - u_{0\alpha}\partial_{0\alpha} u_{0i} = -\frac{1}{\rho}(\partial_i p - \partial_{0i} p_0) + \nu \partial_\alpha^2 u_i - \nu \partial_{0\alpha}^2 u_{0i}, \quad (3)$$

where $\delta u_i = u_i - u_{0i}$ is the velocity increment. In this paper, repeated indices imply summation and no summation is implied by repeated Greek indices. Since u_i (or p_i) depends only on $\mathbf{x}$, and u_{0i} (or p_{0i}) depends only on $\mathbf{x_0}$, equation (3) is reduced to

$$\partial_t \delta u_i + \delta u_\alpha \partial_\alpha \delta u_i + u_{0\alpha}(\partial_\alpha + \partial_{0\alpha})\delta u_i = -\frac{1}{\rho}(\partial_i + \partial_{0i})\delta p + \nu(\partial_\alpha^2 + \partial_{0\alpha}^2)\delta u_i. \quad (4)$$

Here, ∂_α means $\partial/\partial x_\alpha$ and ∂_α^2 means $\partial^2/\partial x_\alpha^2$. After multiplying both sides of this equation by $2\delta u_i$, and averaging, "Monin's" equation follows

$$\frac{\partial}{\partial r_\alpha} D_{ii\alpha} = 2\nu \frac{\partial^2}{\partial r_\alpha^2} D_{ii} - 4\epsilon, \quad (5)$$

where the pressure term $\langle \delta u_i \partial_i \delta p \rangle \equiv -\langle \delta p \partial_i \delta u_i \rangle$ vanishes because of continuity. Because of homogeneity, $\langle u_{0\alpha}\partial_{0\alpha}(\delta u_i)^2 \rangle$ and $\langle u_{0\alpha}\partial_\alpha(\delta u_i)^2 \rangle$ are also zero. Note that Monin's equation could be more directly obtained by first multiplying equation (3) by δu_j and then multiplying the equation for δu_j by δu_i. Adding the two equations yields

$$\frac{\partial}{\partial r_\alpha} D_{ij\alpha} = 2\nu \frac{\partial^2}{\partial r_\alpha^2} D_{ij} - \frac{4}{3}\epsilon \delta_{ij}, \quad (6)$$

with

$$\epsilon = \nu \left\langle (\nabla_\alpha u_i)^2 \right\rangle \cdot \quad (7)$$

D_{ij} and $D_{ij\alpha}$ as well as ϵ can be replaced by their axisymmetric forms since equation (6) is valid for homogeneous turbulence. Axisymmetric forms of D_{ij} and D_{ijk} are developed in Sections 3 and 4 respectively while the axisymmetric form of ϵ is given in Section 5.

Similar generalizations of Monin's equation in the inertial range have been also given by Frisch [12] and Lindborg [13]:

$$\frac{\partial}{\partial r_\alpha} D_{ii\alpha} = -4\epsilon, \quad (8)$$

The basis of their derivations includes homogeneity and incompressibility. In 1997, Hill [14] used local homogeneity to obtain their generalization (8). Hill also discussed the range of applicability to experiments of Kolmogorov's and Monin's equations. Note that equation (8) is not a generalization of Kolmogorov's $\frac{4}{5}$ law as emphasized by Hill, but a generalization of Monin's equation when the viscous effects are negligible.

Monin's equation is needed in subsequent sections in order to derive formulas for locally axisymmetric turbulence. Indeed, one of the primary objectives of this paper is to examine the statistics and the dynamics of fields which are homogeneous, but neither isotropic nor local isotropic. Since the work of Batchelor [24] and Chandresakhar [25], there have been a few studies on axisymmetric turbulence. George and Hussein [26] introduced the concept of locally axisymmetric turbulence and provided experimental support for some of its consequences. They gave a discussion for the nearly axisymmetric turbulence. Locally axisymmetric turbulence was also examined by Antonia *et al.* [6] in the context of DNS data for a fully developed turbulent channel flow and by Hussein [27] for data in a turbulent plane jet. It would appear that only limited progress has been made on the theory of axisymmetric turbulence. For example, relations between the scalar functions of the second-order structure function tensor, or relations between the scalar functions of the third-order structure function tensor or the axisymmetric equivalent of Kolmogorov's equation are not available. This could limit the applicability of the theory. The objective of this work is to derive this kind of relations which can be tested experimentally.

3 Second-order tensor D_{ij}

In this section, the axisymmetric form of the second-order tensor is derived. Relations between the scalar functions of this tensor are also derived. Figure 1 shows the system of orthogonal unit vectors (λ, e_2, e_1) chosen to represent axisymmetric tensors for velocity structure functions. The procedure is similar to that established by Lindborg for the second-order correlation tensor and yields equivalent

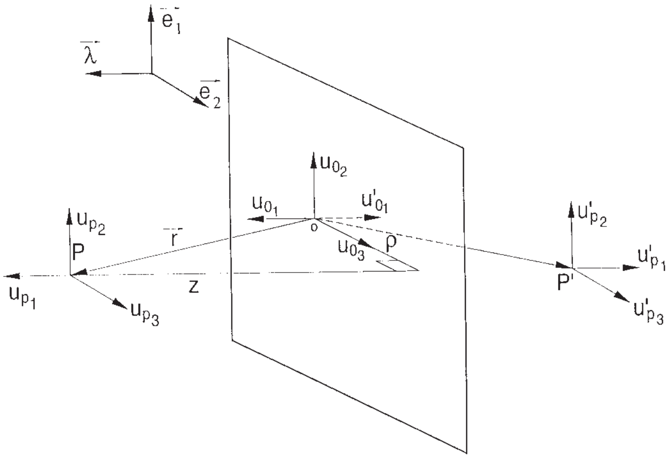

Fig. 1. Cartesian coordinate system showing velocity components.

results. Details are not presented in this section and can be found in reference [18] or [16]. Hereafter, only important results are given. The second-order structure function tensor, in the system of Figure 1, is written

$$D_{ij}(\mathbf{r}) = \lambda_i\lambda_j D_1 + e_{2i}e_{2j}D_2 + e_{1i}e_{1j}D_3 + (\lambda_i e_{2j} + \lambda_j e_{2i})D_4 \quad (9)$$

where the scalar functions D_1, D_2, D_3 and D_4 depend on $\rho = |\mathbf{r}\times\lambda|$ and $z = \mathbf{r}\cdot\lambda$, and can all be measured since they are defined as follows

$$D_1 = \langle(\delta u_1)^2\rangle = \langle(u_{p1} - u_{01})^2\rangle \quad (10a)$$

$$D_2 = \langle(\delta u_2)^2\rangle = \langle(u_{p2} - u_{02})^2\rangle \quad (10b)$$

$$D_3 = \langle(\delta u_3)^2\rangle = \langle(u_{p3} - u_{03})^2\rangle \quad (10c)$$

$$D_4 = \langle\delta u_1 \delta u_2\rangle = \langle(u_{p1} - u_{01})(u_{p2} - u_{02})\rangle \cdot \quad (10d)$$

Note that, in this paper, the longitudinal velocity component u_1 is also written u_L in the case of isotorpic turbulence. The other two components u_2 and u_3 are written u_N for isotropic turbulence. Also, D_1, D_2, D_3 are even in z while D_4 is odd with respect to z.

Relations between D_1, D_2, D_3, D_4 follow from continuity, *i.e.*

$$\frac{\partial}{\partial\rho}(\rho D_4) + \rho\frac{\partial}{\partial z}(D_1) = 0 \quad (11a)$$

$$D_3 = \frac{\partial}{\partial\rho}(\rho D_2) + \rho\frac{\partial}{\partial z}D_4. \quad (11b)$$

The axisymmetric scalar functions D_1, D_2, D_3, D_4 are related to the isotropic ones, D_{LL} and D_{NN}, by

$$D_1 = \frac{z^2}{r^2}D_{LL} + \frac{\rho^2}{r^2}D_{NN} \quad (12a)$$

$$D_2 = \frac{\rho^2}{r^2}D_{LL} + \frac{z^2}{r^2}D_{NN} \quad (12b)$$

$$D_3 = D_{NN} \quad (12c)$$

$$D_4 = \frac{\rho z}{r^2}(D_{LL} - D_{NN}). \quad (12d)$$

After substituting (12 a–d) into equations (11a, b), the isotropic result

$$D_{NN}(r) = \left(1 + \frac{r}{2}\frac{\partial}{\partial r}\right)D_{LL}(r) \quad (13)$$

is obtained. Note that when $\mathbf{r}$ is parallel to λ, *i.e.* in the particular axisymmetric case with $\rho = 0$, we have $D_2 = D_3$ and $D_4 = 0$; this leads to

$$D_{ij}(\mathbf{r}) = \lambda_i\lambda_j D_1 + D_2(\delta_{ij} - \lambda_i\lambda_j) \quad (14)$$

which is similar in form to the second-order isotropic tensor

$$D_{ij}(\mathbf{r}) = \frac{r_i r_j}{r^2}D_{LL} + D_{NN}\left(\delta_{ij} - \frac{r_i r_j}{r^2}\right) \quad (15)$$

However, D_1 and D_2 are independent scalar functions whereas D_{LL} and D_{NN} are related through (13).

Understanding the structure in space of a turbulent flow, as well as its statistical properties, remains a challenge for both the experimentalist and the theoretician. The theory of turbulence have been developed with some success, for the special case of isotropic turbulence. However, from an experimental point of view, isotropic turbulence does not occur naturally, nor is it particularly easy to produce. Many investigations have showed that their results are in disagreement with the classical picture of turbulence which assumes the small scales to be isotropic. For example, it has been known that there are deviations from K41 scaling laws. Namely, the nth order velocity structure function $\langle(\delta u_1)^n\rangle$ does not scale as $r^{n/3}$. Recent wind tunnel experiments [28] and numerical simulations [29] have confirmed this anomalous scaling with a new additional result: scaling exponents (for n being even) measured along the longitudinal direction differed from those in the lateral direction significantly.

In order to assess the isotropy/anisotropy condition of the three dimensional, incompressible, unforced turbulent flow, and using high order resolution simulations, Boratav has computed the second order lateral structure function D_{NN} directly, and compared it to the values of D_{NN} obtained from the isotropic relation equation (13). The computed D_{NN} and the isotropic relation are in good agreement, with some deviations at the larger r end of the inertial range.

Grossmann *et al.* [30] employed Batchelor's parametrization to be able to scale up the longitudinal second order structure function $D_{LL}(r)$ to much larger Reynolds number and thereby get consistent data for the transverse second order structure function and for the third order longitudinal structure function $D_{LLL}(r)$, which for isotropic, homogeneous, incompressible turbulence both follow from $D_{LL}(r)$, namely through the relation equation (13) and through equation (1) respectively. For the transverse second order structure function, they found poor agreement between the curve evaluated from equation (13) and their numerical results. They outlined that the reason is that the flow is not isotropic and homogeneous at the large scales.

As noted by Herweijer and Van de Water [31], the longitudinal and transverse scaling exponents should strictly be equal for homogeneous isotropic turbulence since D_{LL} and D_{NN} are linked through equation (13). However, these scaling exponents are different. The observation that the transverse velocity increments scales differently from longitudinal velocity increments is currently receiving a great deal of attention [32–35]. The difference in scaling, if suitably corroborated, would need to be taken into account in small scale turbulence models. Antonia *et al.* [8] indicated that the transverse velocity increments show larger departures than longitudinal increments from predictions of Kolmogorov (K41). They have pointed out that a source of the discrepancy may be that, for their small value of the Reynolds number R_λ, any anisotropy in the flow will tend to affect the transverse increments more than the longitudinal ones. As R_λ increases and/or the isotropy improves, it is expected that the inequality will eventually disappear.

Remember that the basic notion underlying Kolmogorov's hypothesis is that of local isotropy, which implies that small scales are statistically homogeneous and isotropic. At the level of second order statistics ($\langle(\delta u_i)^n\rangle, n = 2$), the evidence (see Monin and Yaglom [10]) in favour of Kolmogorov's hypothesis is considered so solid that it is often forgotten that there is room for worry [36]. Conclusions from second order statistics are not comforting. For example, local isotropy implies that

$$E_2(k_1) = E_3(k_1) = \frac{1}{2}\left[E_1(k_1) - k_1\frac{\partial E_1(k_1)}{\partial k_1}\right]$$

where E_1, E_2 and E_3 are the spectral densities of the fluctuations in directions 1, 2 and 3 respectively, and k_1 is the component of the wavenumber in the direction 1. The most detailed test of this equation appears to have been made by Champagne [37], who states that the computed spectra $E_2(k_1)$ and $E_3(k_1)$ are in fair agreement with experiment, though consistently higher in the dissipative wavenumber region. This is not an overwhelming endorsement. The few existing measurements of $E_2(k_1)$ and $E_3(k_1)$ – for example Laufer [38] in the pipe flow, Klebanoff [39] and Mestayer [40] in the boundary layer, Kistler and Vrebalovich [41] in grid turbulence – show hardly any 5/3 region, and almost all of them are unsupportive of the result from local isotropy that the ratio $\frac{3E_2(k_1)}{4E_1(k_1)}$ should be unity in the inertial range. The Reynolds numbers in most of these flows have been thought to be respectively high.

The lapses of local isotropy have been voiced before [42,43]. Local isotropy appears a doubtful proposition, at least in the inertial range and for most Reynolds numbers of practical interest. Therefore local axisymmetry is the best alternative to describe the fine scale structures. It is evident that the relations derived in this section for the second order structure functions must play in the theory of locally axisymmetric homogeneous turbulence the same role which the corresponding relations (Eqs. (13, 15)) have played in the theory of locally isotropic homogeneous turbulence. They should help the experimentalist to better understand the structure of a real turbulent flow.

4 Third-order tensor D_{ijk}

In this section, we have developed the two-point representation for the third-order tensor and established the properties of the corresponding scalar functions that are needed for reduction of the general formulae in Section 6. For axisymmetric turbulence without rotation about the axis of symmetry, we can write the third order structure function tensor

$$\begin{aligned} D_{ijk}(\mathbf{r}) = \lambda_i\lambda_j\lambda_k T_1 &\\ &+ T_2(\lambda_j e_{2i}e_{2k} + \lambda_i e_{2j}e_{2k} + \lambda_k e_{2i}e_{2j})\\ &+ T_3(\lambda_j e_{1i}e_{1k} + \lambda_i e_{1j}e_{1k} + \lambda_k e_{1i}e_{1j})\\ &+ T_4(\lambda_i\lambda_k e_{2j} + \lambda_i\lambda_j e_{2k} + \lambda_j\lambda_k e_{2i})\\ &+ e_{2i}e_{2j}e_{2k}T_5\\ &+ T_6(e_{1i}e_{1k}e_{2j} + e_{1i}e_{1j}e_{2k} + e_{1j}e_{1k}e_{2i}) \end{aligned} \quad (16)$$

where T_1, $T_2\cdots T_6$ are scalar functions of ρ and z. This tensor is symmetric in its three indices and it can readily be shown that

$$\frac{\partial^3 D_{ijk}}{\partial r_i\partial r_j\partial r_k} = 0. \quad (17)$$

Now, using definition (16) of the tensor D_{ijk}, equation (17) leads to a relation between the six scalar functions T_1, $T_2\cdots T_6$, *viz.*

$$\begin{aligned} \frac{\partial^3 T_1}{\partial z^3} - \frac{3}{\rho}\,\frac{\partial^2}{\partial\rho\partial z}T_2 + \frac{3}{\rho}\,\frac{\partial^3}{\partial\rho^2\partial z}(\rho T_3)&\\ + \frac{3}{\rho}\,\frac{\partial^3}{\partial z^2\partial\rho}(\rho T_4) + \frac{1}{\rho}\,\frac{\partial^3}{\partial\rho^3}(\rho T_5)&\\ - \frac{3}{\rho}\,\frac{\partial}{\partial\rho}\left(\frac{1}{\rho} + \frac{\partial}{\partial\rho}\right)T_6 = 0& \end{aligned} \quad (18)$$

where

$$\left.\begin{aligned} T_1(\rho,z) &= \langle(\delta u_1)^3\rangle \\ T_2(\rho,z) &= \langle(\delta u_2)^2\delta u_1\rangle \\ T_3(\rho,z) &= \langle(\delta u_3)^2\delta u_1\rangle \\ T_4(\rho,z) &= \langle(\delta u_1)^2\delta u_2\rangle \\ T_5(\rho,z) &= \langle(\delta u_2)^3\rangle \\ T_6(\rho,z) &= \langle(\delta u_3)^2\delta u_2\rangle \end{aligned}\right\}. \tag{19}$$

Equation (18) shows that the six scalar functions are related by a single expression. This relation is similar to the two relations, equations (11a, b), corresponding to the second-order tensor $D_{ij}(\mathbf{r})$.

As in Section 3, it is possible to establish relations between the axisymmetric scalar functions T_1, T_2, $\cdots T_6$ and the isotropic third-order functions D_{LNN} and D_{LLL}. The third-order isotropic tensor can be expressed in terms of the two non-vanishing scalar functions

$$D_{ijk}(\mathbf{r}) = (D_{LLL} - 3D_{LNN})\frac{r_i r_j r_k}{r^3} + D_{LNN}\left(\frac{r_i}{r}\delta_{jk} + \frac{r_j}{r}\delta_{ik} + \frac{r_k}{r}\delta_{ij}\right). \tag{20}$$

Now, we project this tensor onto the tensors $\lambda_i\lambda_j\lambda_k$, $\lambda_k e_{2i}e_{2j}$, $\lambda_k e_{1i}e_{1j}$, $\lambda_i\lambda_j e_{2k}$, $e_{2i}e_{2j}e_{2k}$ and $e_{2i}e_{1j}e_{1k}$ which corresponds to T_1, T_2, T_3, T_4, T_5 and T_6 respectively. The resulting formulas for the T_i's (which give relations between the axisymmetric functions and the isotropic ones) are as follows

$$T_1 = (D_{LLL} - 3D_{LNN})\frac{z^3}{r^3} + D_{LNN}\left(\frac{3z}{r}\right) \tag{21a}$$

$$T_2 = (D_{LLL} - 3D_{LNN})\frac{z\rho^2}{r^3} + D_{LNN}\left(\frac{z}{r}\right) \tag{21b}$$

$$T_3 = \frac{z}{r}D_{LNN} \tag{21c}$$

$$T_4 = (D_{LLL} - 3D_{LNN})\frac{z^2\rho}{r^3} + D_{LNN}\left(\frac{\rho}{r}\right) \tag{21d}$$

$$T_5 = (D_{LLL} - 3D_{LNN})\frac{\rho^3}{r^3} + D_{LNN}\left(\frac{3\rho}{r}\right) \tag{21e}$$

$$T_6 = \frac{\rho}{r}D_{LNN}. \tag{21f}$$

Note that when λ and $\mathbf{r}$ are parallel, *i.e.* in the particular axisymmetric case with $\rho = 0$, we have $T_2 = T_3$ and $T_4 = T_5 = T_6 = 0$. In this case, the representation of the tensor becomes similar to the representation of the isotropic one (here Eq. (20))

$$D_{ijk}(\mathbf{r}) = \lambda_i\lambda_j\lambda_k(T_1 - 3T_2) + T_2(\lambda_j\delta_{ik} + \lambda_i\delta_{jk} + \lambda_k\delta_{ij}). \tag{22}$$

Of course, the isotropic functions D_{LLL} and D_{LNN} are related through (25), whereas T_1 et T_2 are axisymmetric independent scalar functions.

With the help of relations (21a–f), we can verify that equation (18) is reduced, in the case of isotropic turbulence, to the well-known relation between the isotropic scalar functions of the tensor $D_{ijk}(\mathbf{r})$. After substituting (21 a–f) into equation (18) and using the definitions $r^2 = \rho^2 + z^2$, $\rho = r(1-\mu^2)^{1/2}$, $z = r\mu$, it can be shown that

$$\frac{6}{r^2}\left(\frac{\partial}{\partial r} + r\frac{\partial^2}{\partial r^2} + \frac{r^2}{6}\frac{\partial^3}{\partial r^3}\right)D_{LLL} = \frac{18}{r^2}\left(\frac{\partial}{\partial r} + \frac{r}{3}\frac{\partial^2}{\partial r^2}\right)D_{LNN}. \tag{23}$$

This can be re-written

$$\left(1 + \frac{r}{3}\frac{\partial}{\partial r}\right)\frac{\partial}{\partial r}\left[\frac{1}{6}\left(1 + r\frac{\partial}{\partial r}\right)D_{LLL} - D_{LNN}\right] = 0. \tag{24}$$

Using Monin and Yaglom's argument (p. 122), the only solution of (24) which does not have a singularity at $r = 0$, is

$$D_{LNN} = \frac{1}{6}\left(1 + r\frac{\partial}{\partial r}\right)D_{LLL}, \tag{25}$$

i.e. the isotropic result [10].

In the theory of locally isotropic turbulence, particular attention was given to the second order structure function, *i.e.* the averaged squared values of the velocity increment. The implication of the theory of local isotropy with regard to the behavior of the above structure function, and of the third order structure function, and more generally, the moments of spatial velocity increments are well known [10]. The most typical statistics studied for the approach to local isotropy are the ratios of the off-diagonal components of the velocity spectrum tensor to its on-diagonal components. These statistics are studied by Saddoughi and Veeravalli [44] and Borue and Orszag [45] as well as by others. In terms of structure functions, the results is that for $\alpha \neq \beta$, $D_{\alpha\beta}(r)/D_{11}(r)$ decreases proportional to $r^{2/3}$ in the inertial range (a consequence of K41). The fact that the second order structure function follows an $r^{2/3}$ law implies a $k^{-5/3}$ law for the energy spectrum. Experimental results sometimes support the K41 theory as far as the second order structure function (and thus the spectrum) is concerned. The consistency between the K41 theory and experimental data on structure functions is questionable when $n > 3$ [12].

The third order structure function tensor and relations between its scalar functions have been sometimes employed to check local isotropy. Hill and Thorodsen [46] gave formulas for the two-point correlation of fluid-particle acceleration in terms of velocity structure functions. The two-point correlation of the fluid-particle acceleration is the sum of pressure gradient and viscous force correlations. The pressure gradient correlation is related to the fourth order velocity structure function (for the assumption of joint Gaussian velocities, the pressure gradient contribution to acceleration correlation is linked to the second order structure function). The acceleration correlation caused by viscous forces is formulated in terms of the third order velocity structure function (the incompressibility condition, Eq. (25), is used). Velocity data from grid-generated turbulence in a wind tunnel are used to evaluate these quantities. The evaluated relationships require only the Navier-Stokes equation, incompressibility, local homogeneity and local isotropy. A motivation of these authors is to encourage the design of experiments capable of evaluating the acceleration correlation.

Frisch [12] re-expressed the energy flux in terms of third order moments of longitudinal velocity increments. These are much simpler to measure experimentally. However relations between the third order scalar functions are not usually used to study the approach to local isotropy/anisotropy. An experience with nearly homogeneous wind-tunnel grid turbulence suggest that equation (25) is not easily satisfied even when equation (13) is satisfied [14].

5 Dissipation

There is a considerable evidence that local isotropy is not an adequate description of the velocity derivatives moments for at least the finite Reynolds numbers associated with many turbulent laboratory flows (Georges and Hussein [26]).

The dissipation term in equation (5) is defined, for homogeneous turbulence, by equation (7). For isotropic turbulence, this equation is reduced to

$$\epsilon_{\text{iso}} = 15\nu \left\langle \left(\frac{\partial u_1}{\partial x_1}\right)^2 \right\rangle \cdot \tag{26}$$

The general form of ϵ in homogeneous turbulence is (Batchelor [24]; George and Hussein [26])

$$\epsilon_{ij} = -\nu \left(\frac{\partial^2 B_{ij}(\mathbf{r})}{\partial r_n \partial r_m}\right)_{r=0} = \nu \left\langle \frac{\partial u_i}{\partial x_m}\frac{\partial u_j}{\partial x_n} \right\rangle \tag{27a}$$

or,

$$\begin{aligned}\epsilon = 2\nu \Bigg[&\left\langle \left(\frac{\partial u_1}{\partial x_1}\right)^2 \right\rangle + \left\langle \left(\frac{\partial u_2}{\partial x_2}\right)^2 \right\rangle \\ &+ \left\langle \left(\frac{\partial u_3}{\partial x_3}\right)^2 \right\rangle + \left\langle \frac{\partial u_1}{\partial x_2}\frac{\partial u_2}{\partial x_1} \right\rangle \\ &+ \left\langle \frac{\partial u_1}{\partial x_3}\frac{\partial u_3}{\partial x_1} \right\rangle + \left\langle \frac{\partial u_2}{\partial x_3}\frac{\partial u_3}{\partial x_2} \right\rangle \Bigg] \\ &+ \nu \Bigg[\left\langle \left(\frac{\partial u_1}{\partial x_2}\right)^2 \right\rangle + \left\langle \left(\frac{\partial u_2}{\partial x_1}\right)^2 \right\rangle \\ &+ \left\langle \left(\frac{\partial u_1}{\partial x_3}\right)^2 \right\rangle + \left\langle \left(\frac{\partial u_3}{\partial x_1}\right)^2 \right\rangle \\ &+ \left\langle \left(\frac{\partial u_2}{\partial x_3}\right)^2 \right\rangle + \left\langle \left(\frac{\partial u_3}{\partial x_2}\right)^2 \right\rangle \Bigg] \cdot \end{aligned} \tag{27b}$$

For axisymmetric turbulence, in the (λ_i, e_{2i}, e_{1i}) coordinate system, equation (27a) becomes when $i = j$

$$\epsilon = -\nu \left[\left(\frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial \rho^2} + \frac{\partial^2}{\partial z^2}\right) B_{ii}(\mathbf{r}) \right]_{r=0}, \tag{28}$$

where $B_{ii}(\mathbf{r}) = B_1 + B_2 + B_3$. Now, we can write Taylor expansions for B_1, B_2 and B_3

$$\begin{aligned}B_\alpha = B_{0\alpha} + a_\alpha \rho^2 + b_\alpha z^2 + c_\alpha \rho^4 \\ + d_\alpha z^4 + e_\alpha \rho^2 z^2 + ...\end{aligned}$$

where $\alpha = 1$, 2 or 3. Using these expansions into (28), we have showed that [16]

$$\epsilon = 4\nu(a_1 + a_2 + a_3) + 2\nu(b_1 + b_2 + b_3) \tag{29}$$

where

$$a_\alpha = \frac{1}{2!}\left(\frac{\partial^2 B_\alpha}{\partial \rho^2}\right)_{r=0}, \quad b_\alpha = \frac{1}{2!}\left(\frac{\partial^2 B_\alpha}{\partial z^2}\right)_{r=0}.$$

The a_α's and b_α's can be equally written

$$a_1 = \frac{1}{2}\left\langle \left(\frac{\partial u_1}{\partial x_2}\right)^2 \right\rangle, \quad a_2 = \frac{1}{2}\left\langle \left(\frac{\partial u_2}{\partial x_2}\right)^2 \right\rangle,$$

$$a_3 = \frac{1}{2}\left\langle \left(\frac{\partial u_3}{\partial x_2}\right)^2 \right\rangle, \quad b_1 = \frac{1}{2}\left\langle \left(\frac{\partial u_1}{\partial x_1}\right)^2 \right\rangle,$$

$$b_2 = \frac{1}{2}\left\langle \left(\frac{\partial u_2}{\partial x_1}\right)^2 \right\rangle, \quad b_3 = \frac{1}{2}\left\langle \left(\frac{\partial u_3}{\partial x_1}\right)^2 \right\rangle.$$

Therefore, the axisymmetric form of the dissipation is

$$\begin{aligned}\epsilon_{\text{axi}} = 2\nu \Bigg[&\left\langle \left(\frac{\partial u_1}{\partial x_2}\right)^2 \right\rangle + \left\langle \left(\frac{\partial u_2}{\partial x_2}\right)^2 \right\rangle \\ &+ \left\langle \left(\frac{\partial u_3}{\partial x_2}\right)^2 \right\rangle \Bigg] + \nu \Bigg[\left\langle \left(\frac{\partial u_1}{\partial x_1}\right)^2 \right\rangle \\ &+ \left\langle \left(\frac{\partial u_2}{\partial x_1}\right)^2 \right\rangle + \left\langle \left(\frac{\partial u_3}{\partial x_1}\right)^2 \right\rangle \Bigg] \cdot \end{aligned} \tag{30}$$

Moreover, in the particular case of axisymmetric turbulence with $\mathbf{r}$ parallel to λ, since $B_2 = B_3$, we have $b_2 = b_3$ and $a_3 = 3a_2 - b_1$. In this case, the dissipation when $\mathbf{r}$ is parallel to λ is reduced to

$$\epsilon = 4\nu(a_1 + 4a_2) + 2\nu(-b_1 + 2b_2) \tag{31}$$

or

$$\begin{aligned}\frac{\epsilon_{\text{axi}}}{\nu} = -\left\langle \left(\frac{\partial u_1}{\partial x_1}\right)^2 \right\rangle + 8\left\langle \left(\frac{\partial u_2}{\partial x_2}\right)^2 \right\rangle \\ + 2\left\langle \left(\frac{\partial u_1}{\partial x_2}\right)^2 \right\rangle + 2\left\langle \left(\frac{\partial u_2}{\partial x_1}\right)^2 \right\rangle\end{aligned} \tag{32}$$

which is identical to the dissipation derived by Georges and Hussein [26]. These two equations for the dissipation (30) and (32) are reduced to the well-known dissipation (26) when axisymmetry turns into isotropy.

The characterization and measurement of small-scale motions of turbulent flow have presented two of the most challenging problems of turbulence research over the past 50 years. One of the primary experimental concerns has been the determination of the rate of dissipation of turbulent kinetic energy. Direct measurement of the average dissipation clearly requires measurements of various components of the spatial derivatives (see Eq. (27b). Because this is impossible in practice, investigators have usually relied on the assumption of local isotropy and Taylor's frozen-field hypothesis in determining the dissipation. Remember that local isotropy implies that the fine structure of turbulence is isotropic, so that its statistical properties are invariant with respect to rotation and reflection about the coordinate axes. Several kinematics conditions are imposed on the fine structure velocity and temperature fields. Taylor [47] derived simple relations between mean square velocity derivatives at one point in a turbulent flow; these relations are clearly important as they permitted the dissipation of turbulent energy to be written in terms of only one, easily measured, mean square velocity derivative: $\langle u_{1,1}^2 \rangle \cdot$

The assumption of local isotropy simplified the experiments significantly but the accuracy of the results were impaired by the fact that shear flows such as the jet did not usually satisfy the relations for local isotropy. Hussein [27] showed that this flow does not satisfy the requirements of local isotropy. He showed that the flow satisfies the conditions for local axisymmetric turbulence introduced by George and Hussein [26] to within the experimental error. These conditions give relations between the spatial derivatives. From these relations, it follows immediately the axisymmetric form of the dissipation of turbulent energy equation (32).
George and Hussein [26] showed, using their measurements in a relatively high Reynolds number round jet and those of Brown *et al.* [5] in a low Reynolds number wake, that mean square values of velocity derivatives are in quite reasonable agreement with local axisymmetry. Browne *et al.* [5] have shown that the isotropic relation equation (26) underestimates the dissipation by almost 45% at the wake centerline of the cylinder, where the flow is fully turbulent, and by 80% near the edge of the wake, where the effect of intermittency becomes important.

As indicated previously, many experimental and theoretical studies do not support Kolmogorov's theory and the deviations are usually related to intermittency in the energy dissipation. Intermittency is a central problem in turbulence. Many attempts have been made to obtain the numerical values of the intermittency exponent from experiments. Note that the space derivative is generally approximated by the time derivative according to Taylor's frozen flow hypothesis, and that one component of dissipation is considered as an adequate representation of the total dissipation statistically. However, Sreenivasan and Kailasnath [48] indicate that these approximations are not critical to the determination of the intermittency exponent.

Direct numerical simulations of fully developed turbulent channel flow at two low Reynolds numbers show that the average dissipation is not consistent with local isotropy, except near the channel centerline. When the assumption of local isotropy is relaxed to one of local axisymmetry, satisfactory agreement is indicated by the data outside the wall region [6].

These agreements obtained between local axisymmetry and available measurements or direct numerical simulations indicate that a more accurate representation can be obtained for the energy dissipation than by invoking local isotropy. The implication of this result should be significant experimentally and can be used to enhance and improve turbulence models. Most importantly, local axisymmetry places the dissipation and their components (for a real flow) within the reach of the experimentalist. The assumption of local axisymmetry makes it possible to measure the accurate dissipation using relatively simple hot-wire configurations. All the previous relations derived for the second order (see Sect. 3)and third order (see Sect. 4) structure function tensors and for the energy dissipation (see Sect. 5) for locally axisymmetric turbulence are needed in subsequent sections.

6 Axisymmetric form of Monin's equation

To simplify the arithmetic, equation (5), which is valid for homogeneous turbulence, will be used as the starting point. After substituting into this equation the axisymmetric tensors $D_{ii\alpha}(\mathbf{r})$ by equation (16), $D_{ii}(\mathbf{r})$ by equation (9) and the dissipation by equation (30), the result is the general axisymmetric form of Monin' equation

$$\frac{\partial}{\partial z}(T_1+T_2+T_3)+\left(\frac{1}{\rho}+\frac{\partial}{\partial\rho}\right)(T_4+T_5+T_6)= 2\nu\left(\frac{\partial^2}{\partial z^2}+\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)(D_1+D_2+D_3)-4\epsilon_{\rm axi}\,. \quad (33)$$

6.1 Isotropy

It can be readily shown that the axisymmetric form of Monin's equation (33) is reduced to Monin's equation which leads to Kolmogorov's equation in the case of isotropic turbulence (when λ is allowed to assume any direction). Substituting relations (12a–d) and (13) for the second-order tensor and relations (21a–f) and (25) for the third-order tensor into equation (33), we obtain

$$\left(\frac{2}{r}+\frac{\partial}{\partial r}\right)\left[\frac{1}{3}\left(4+r\frac{\partial}{\partial r}\right)D_{LLL} -2\nu\frac{\partial}{\partial r}\left(3+r\frac{\partial}{\partial r}\right)D_{LL}+\frac{4}{3}\epsilon_{\rm iso}r\right]=0. \quad (34)$$

By noting that

$$\frac{\partial}{\partial r}\left(3+r\frac{\partial}{\partial r}\right)\equiv\left(4+r\frac{\partial}{\partial r}\right)\frac{\partial}{\partial r}$$

$$\frac{4}{3}\epsilon_{\rm iso}r\equiv\left(4+r\frac{\partial}{\partial r}\right)\left(\frac{4}{15}\epsilon_{\rm iso}r\right)$$

(39) can be written

$$\left(\frac{2}{r}+\frac{\partial}{\partial r}\right)\left(4+r\frac{\partial}{\partial r}\right)\times\left[\frac{1}{3}D_{LLL}-2\nu\frac{\partial}{\partial r}D_{LL}+\frac{4}{15}\epsilon_{\rm iso}r\right]=0 \quad (35)$$

which is reduced to Kolmogorov's equation (1) by invoking the same argument that was used to derive equation (25).

6.2 Axisymmetry with r parallel to λ

Finally, it is worth noting that in the particular axisymmetric case when **r** is parallel to λ, equation (33) can be somewhat simplified. In this case, $T_2=T_3$, $D_2=D_3$ and $T_4=T_5=T_6=0$. However, although the sum $T_4+T_5+T_6$ is zero, its division by ρ or its derivation with respect to ρ is not zero. This equation for axisymmetric turbulence, valid when **r** is parallel to λ, can be verified by experiment. In particular, in the inertial range, where the viscous effect is negligible, equation (33) becomes

$$\frac{\partial}{\partial z}(T_1+2T_2)+\left(\frac{1}{\rho}+\frac{\partial}{\partial\rho}\right)(T_4+T_5+T_6)=-4\epsilon_{\rm axi}\,. \quad (36)$$

6.3 Kolmogorov's equation

In this subsection, we are deriving the axisymmetric form of Kolmogorov's equation in the particular case when **r** is parallel to λ. First, we are focusing on the properties of the scalar functions of the third-order tensor: (i) reflectional symmetry implies that T_1, T_2 and T_3 are odd in z and even in ρ while T_4, T_5 and T_6 are even in z and odd in ρ; (ii) when **r** is parallel to λ, we have $T_2=T_3$ and $T_4=T_5=T_6=0$.

Consequently, since T_1, T_2 and T_3 are even in ρ, their first-order derivatives with respect to ρ are zero when **r** is parallel to λ. Moreover since T_4, T_5 and T_6 are odd in ρ, their first-order derivatives with respect to z are zero. An other consequence when $\rho=0$ is that the first-order derivative, with respect to ρ, of T_4 (or T_5 or T_6) is equal to the ratio $\frac{T_4}{\rho}$ (or $\frac{T_5}{\rho}$ or $\frac{T_6}{\rho}$ respectively). All these remarks allow us to write equation (36) as follow

$$\left[\left(\frac{\partial}{\partial z}+2\frac{\partial}{\partial\rho}\right)(T_1+2T_2+T_4+T_5+T_6)-4\epsilon_{\rm axi}\right]_{\rho=0}=0. \quad (37)$$

Now, let us find a solution to equation (37). One can note that the dissipation rate is constant. Therefore it is possible to write (37) as follow

$$\left[\left(\frac{\partial}{\partial z}+2\frac{\partial}{\partial\rho}\right)\left(\sum_{i=1}^{6}T_i-\epsilon_{\rm axi}(az+b\rho+c\rho z)\right)\right]_{\rho=0}=0. \quad (38)$$

Since $T_4=T_5=T_6=0$, a solution to equation (38), in the case **r** parallel to λ, is

$$T_1+2T_2=-\frac{4}{3}\epsilon_{\rm axi}z \quad (39)$$

where z is identical to the spacing r since $\rho=0$ $(z=r)$. equation (39) should have more general validity than the four-fifths law for isotropic turbulence, *i.e.* equation (2).

When axisymmetry (with **r** parallel to λ) turns into isotropy, equation (39) is reduced to Kolmogorov's equation. Indeed, equations (21a, b) are reduced to $T_1=D_{LLL}$ and $T_2=D_{LNN}$. The axisymmetric dissipation is reduced to the isotropic one. Using the relation between the third-order isotropic scalar functions D_{LLL} and D_{LNN}, equation (25) and substituting all these relations into (44) yield

$$\frac{1}{3}\left(4+r\frac{\partial}{\partial r}\right)D_{LLL}=-\frac{4}{3}\epsilon_{\rm iso}r \quad (40)$$

or

$$\frac{1}{3}\left(4+r\frac{\partial}{\partial r}\right)\left[D_{LLL}-\frac{4}{5}\epsilon_{\rm iso}r\right]=0. \quad (41)$$

By invoking the argument used to derive equation (25), a solution of equation (41) is the $\frac{4}{5}$ law, *i.e.* equation (2).

Kolmogorov's equation equation (39) is an exact relation between the longitudinal second order and third order structure functions, $\langle(\delta u_1)^2\rangle$ and $\langle(\delta u_1)^3\rangle$, valid for the ideal case of homogeneous isotropic turbulence. The second order structure function, $\langle(\delta u_1)^2\rangle$, is linked to kinetic energy and the third order one, $\langle(\delta u_1)^3\rangle$, is linked to energy transfer, two crucial quantities characterizing fully developed turbulence.

The inertial range law for the third order longitudinal structure function (Eq. (39)) is the only inertial range scaling law that has been derived from the Navier-Stokes equations, and must therefore be considered to be a corner-stone of the theory. The fundamental importance of this law has been pointed out by Frisch [49] and by Hunt and Vassilicos [50], among others. It is considered as the most sensitive and most appropriate test of local isotropy hypothesis [13]. For example, Kolmogorov's $\frac{4}{5}$ law was of practical utility for determining the energy dissipation rate. Since the majority of turbulent energy budgets in the literature have been based on isotropy, they are likely to be in error and any conclusion can only be viewed with reservation. Similarly, any computer models that use estimates of the turbulent energy dissipation based on isotropy, are likely to be in error. Therefore, relation (39) can be used as a sensitive test of the accuracy of the measured dissipation and at the same time the isotropy/anisotropy of the third order structure function tensor. The present theory of local axisymmetric turbulence will clarify the manner in which local isotropic turbulence comes to prevail and will account for the reappearance of anisotropy under certain conditions. The reader can find in Section 7 more details and a discussion on the consequences of the theory of local axisymmetric turbulence and of the corresponding form of Kolmogorov's equation.

6.4 Limiting form of Kolmogorov's equation when r → 0

6.4.1 Second-order scalar functions

In order to derive the limiting form of the axisymmetric form of Monin's equation which is exactly equivalent to the limiting form of the axisymmetric form of Kolmogorov's equation, when $r \to 0$, we use Taylors series expansions about $r = 0$. For small values of r, we deduce the expansions of the second-order scalar functions D_1, D_2, D_3 and D_4. For $\alpha = 1$, 2 or 3, the function D_α has the following general form

$$\begin{aligned} D_\alpha = {} & \rho^2 \left\langle \left(\frac{\partial u_\alpha}{\partial \rho}\right)^2 \right\rangle + z^2 \left\langle \left(\frac{\partial u_\alpha}{\partial z}\right)^2 \right\rangle \\ & + z^2\rho^2 \left[\frac{1}{2} \left\langle \frac{\partial^2 u_\alpha}{\partial \rho^2} \left(\frac{\partial^2 u_\alpha}{\partial z^2}\right)^2 \right\rangle + \left\langle \left(\frac{\partial^2 u_\alpha}{\partial \rho \partial z}\right)^2 \right\rangle \right. \\ & \left. + \left\langle \frac{\partial u_i}{\partial \rho} \frac{\partial^3 u_\alpha}{\partial \rho \partial z^2} \right\rangle + \left\langle \frac{\partial u_\alpha}{\partial z} \frac{\partial^3 u_\alpha}{\partial \rho^2 \partial z} \right\rangle \right] \\ & + \rho^4 \left[\frac{1}{4} \left\langle \left(\frac{\partial^2 u_\alpha}{\partial \rho^2}\right)^2 \right\rangle + \frac{1}{3} \left\langle \frac{\partial u_\alpha}{\partial \rho} \frac{\partial^3 u_\alpha}{\partial \rho^3} \right\rangle \right] \\ & + z^4 \left[\frac{1}{4} \left\langle \left(\frac{\partial^2 u_\alpha}{\partial z^2}\right)^2 \right\rangle + \frac{1}{3} \left\langle \frac{\partial u_\alpha}{\partial z} \frac{\partial^3 u_\alpha}{\partial z^3} \right\rangle \right] + ... \end{aligned} \tag{42}$$

We can also simply write this general form for $\alpha = 1$, 2 or 3

$$D_\alpha = a_\alpha \rho^2 + b_\alpha z^2 + c_\alpha \rho^4 + d_\alpha z^4 + e_\alpha \rho^2 z^2 + ... \tag{43}$$

and

$$D_4 = a_4 z\rho + b_4 z\rho^3 + c_4 \rho z^3 + ... \tag{44}$$

The continuity relations (11a–b) allow us to reduce the number of coefficients in the previous expansions. For the particular case $\rho = 0$, we find

$$\begin{aligned} & a_4 = -b_1, && b_4 = -\frac{1}{2}e_1, && c_4 = -2d_1, \\ & c_3 = 5c_2 + b_4, && a_3 = 3a_2 + a_4, && \\ & b_2 = b_3, && d_2 = d_3, && e_3 = 3e_2 + 3c_4. \end{aligned} \tag{45}$$

In the isotropic case, we can note that the two scalar functions D_{LL} and D_{NN} are related to B_{LL} and B_{NN}

$$\begin{aligned} D_{LL}(r) &= 2\left[B(0) - B_{LL}(r)\right], \\ D_{NN}(r) &= 2\left[B(0) - B_{NN}(r)\right]. \end{aligned} \tag{46}$$

Moreover, the expansions of the isotropic functions f and g are

$$f = \frac{B_{LL}(r)}{B_{LL}(0)} = 1 - \frac{r^2}{2\lambda_t^2} + \frac{\alpha}{4!} r^4 + ... \tag{47}$$

$$g = \frac{B_{NN}(r)}{B_{NN}(0)} = 1 - \frac{r^2}{\lambda_t^2} + \frac{3\alpha}{4!} r^4 + ... \tag{48}$$

where λ_t is the Taylor microscale and $B_{LL}(0) = B_{NN}(0) = u^2$, u being the root mean square of any velocity components. Therefore, we can express the expansions of the structure functions D_{LL} and D_{NN} using the expansions of f and g

$$D_{LL}(r) = -2u^2 \left(1 - \frac{r^2}{2\lambda_t^2} + \frac{\alpha}{4!} r^4 + ...\right) \tag{49}$$

$$D_{NN}(r) = -2u^2 \left(1 - \frac{r^2}{\lambda_t^2} + \frac{3\alpha}{4!} r^4 + ...\right). \tag{50}$$

Now, when axisymmetry turns into isotropy, equations (12a–d) yield to

$$\begin{aligned} & a_1 = 2a_2 = a_3 = 2b_1 = b_2 = b_3 = -\frac{u^2}{\lambda^2} \\ & c_1 = 3c_2 = c_3 = 3d_1 = d_2 = d_3 = -\frac{\alpha}{8} u^2 \\ & 6e_1 = 6e_2 = 4e_3 = \alpha u^2. \end{aligned} \tag{51}$$

6.4.2 Third-order scalar functions

The behaviour of the third-order scalar functions T_1, T_2, ..., T_6, for small values of r, is given by the following expansions for $i = 1$, 2 or 3

$$\begin{aligned} T_i = {} & z^3 \left\langle \frac{\partial u_1}{\partial z} \left(\frac{\partial u_\alpha}{\partial z}\right)^2 \right\rangle + \left[\left\langle \frac{\partial u_1}{\partial z} \left(\frac{\partial u_\alpha}{\partial \rho}\right)^2 \right\rangle \right. \\ & \left. + 2 \left\langle \frac{\partial u_1}{\partial z} \frac{\partial u_\alpha}{\partial \rho} \frac{\partial u_\alpha}{\partial z} \right\rangle \right] z\rho^2 + ... \end{aligned} \tag{52}$$

where $i = 1$, 2 or 3 corresponds to $\alpha = 1$, 2 or 3 respectively. We can equivalently write in a simple manner

$$T_i = \alpha_i z^3 + \beta_i z\rho^2 + ... \tag{53}$$

Next, for $i = 4$, 5 or 6

$$\begin{aligned} T_i = {} & \rho^3 \left\langle \frac{\partial u_2}{\partial \rho} \left(\frac{\partial u_\alpha}{\partial \rho}\right)^2 \right\rangle + \left[\left\langle \frac{\partial u_2}{\partial \rho} \left(\frac{\partial u_\alpha}{\partial z}\right)^2 \right\rangle \right. \\ & \left. + 2 \left\langle \frac{\partial u_2}{\partial z} \frac{\partial u_\alpha}{\partial z} \frac{\partial u_\alpha}{\partial \rho} \right\rangle \right] \rho z^2 + ... \end{aligned} \tag{54}$$

where $i = 4$, 5 or 6 is relative to $\alpha = 1$, 2 or 3 respectively. We can also simply write

$$T_i = \zeta_\alpha \rho z^2 + \gamma_\alpha \rho^3 + ... \tag{55}$$

In the case $\rho = 0$, we have $\alpha_2 = \alpha_3$. For small values of r, the expansion of the longitudinal isotropic structure function is

$$D_{LLL}(r) = \langle u_{1,1}^3 \rangle r^3 + ... \tag{56}$$

Combining equations (53, 55, 56) (21a–f) and (25) leads to

$$\alpha_1 = \frac{3}{2}\alpha_2 = \frac{3}{2}\alpha_3 = \langle u_{1,1}^3 \rangle$$
$$-6\zeta_4 = \zeta_5 = 3\zeta_6 = 2\langle u_{1,1}^3 \rangle\cdot \tag{57}$$

Note that

$$\langle u_{1,1}^3 \rangle = B_{LL,L}^{(3)}(0) = 6b_3 = 6\tau u^3. \tag{58}$$

6.4.3 The limiting form

We can easily derive the limiting form of the axisymmetric form of Kolmogorov's equation using the previous expansions of the second and third order tensors. We can substitute into the general axisymmetric form of Monin's equation (33), the Taylor expansions of the scalar functions T_1, ..., T_6, equations (53) and (55), and the expansions of the scalar functions D_1, ... D_3, equations (43), as well as the axisymmetric form of the dissipation rate (32). The result is the limiting form of Kolmogorov's equation for axisymmetric turbulence when $\rho = 0$

$$3\sum_{i=1}^{3}\alpha_i + 2\sum_{i=1}^{3}\zeta_i = 8\nu\sum_{i=1}^{3}e_i + 24\nu\sum_{i=1}^{3}d_i. \tag{59}$$

In view of the isotropic relations (51) and (57), the result (59) is reduced to the well-known limiting form of Kolmogorov's equation when axisymmetry turns into isotropy

$$\left\langle \left(\frac{\partial u_1}{\partial x_1}\right)^3 \right\rangle = -2\nu \left\langle \left(\frac{\partial^2 u_1}{\partial x_1^2}\right)^2 \right\rangle,$$

i.e. the isotropic form of the vorticity budget [51,52].

The term $\langle u_{1,11}^2 \rangle$ is linked to the mean square vorticity $\langle \omega_{1,2}^2 \rangle$ [53], *i.e.*:

$$\langle u_{1,11}^2 \rangle = \frac{3}{14}\langle \omega_{1,11}^2 \rangle \cdot$$

The third order moment $\langle u_{1,1}^3 \rangle$ is equal to

$$\langle u_{1,1}^3 \rangle = -\frac{2}{35}\left\langle \omega_i\omega_k\frac{\partial u_i}{\partial x_k}\right\rangle;$$

it describes the production of the vorticity by a stretching of the vortex line. The assumption of isotropy greatly simplifies the discussion of turbulence and many results have been obtained. The most important result is Kolmogorov's equation and its limiting form. However the classical picture of small scales in turbulence being locally isotropic and universal is certainly questionable. Thus, the most readily verifiable consequences of local axisymmetry are the conclusion that the inertial range Kolmogorov's law has the axisymmetric form of equation (39). It is also easy to verify the implied relations between the squares of space derivatives of the velocity $\langle u_{1,1}^3 \rangle$ and $\langle u_{1,11}^2 \rangle$, *i.e.* the limiting form (Eq. (59)). The axisymmetric equivalent of the $\frac{4}{5}$ law is one of the most important results for fully developed turbulence because it is both exact and non trivial. The limiting form (Eq. (59)) of Kolmogorov's equation for locally axisymmetric and homogeneous turbulence is also an important result. It should be the vorticity budget for locally axisymmetric turbulence. The vorticity budget is essential to understanding turbulence dynamics. Thus, these results constitute a kind of "boundary condition" on the theory of turbulence. All the previous remarks suggest that further investigations of kinematics and dynamics for locally axisymmetric turbulence should lead to a useful insight into the fine structure of turbulence.

7 Consequences

Dissipation rate

According to K41, the structure functions obey the relation $\langle(\delta u_1)^3\rangle = C_n r^{n/3}\langle\epsilon\rangle^{n/3}$, where C_n are universal constants. Of these structure function relations, an exact relation is known only for the third-order. This exact relation is the $\frac{4}{5}$ law of Kolmogorov, equations (2). This law has been widely used by experimentalists to determine the scaling range. It was also used to estimate the dissipation rate ϵ. But there have been few attempts in the literature to determine ϵ directly from this law. Indeed, the isotropic relation $\epsilon_{\rm iso} = 15\nu\langle u_{1,1}^2\rangle$ was generally assumed to evaluate the energy dissipation rate. Many experimentalists have measured this so-called pseudo- dissipation rate. This was done instead of measuring (Eq. (27b)) the energy dissipation rate ϵ directly because it is hard to measure all the simultaneous components of the strain tensor. In most cases, it is to be remarked that several current important ideas on the fine structure of turbulence have been established by analysis based on $\epsilon_{\rm iso}$. In particular, it is generally believed that a good indicator of the scaling range is provided by the behavior of $\langle(\delta u_1^*)^3\rangle$, namely

$$\langle(\delta u_1^*)^3\rangle = -\frac{4}{5}r^* \tag{60}$$

the $\frac{4}{5}$ law. The superscript * denotes normalization by the Kolmogorov length scale η and Kolmogorov's velocity scale $v_K = (\nu\epsilon_{\rm iso})^{1/4}$.

However, the true energy dissipation rate ϵ may give different results. This possibility has already been pointed out [54,55]. In the last reference, the authors revealed important differences between the true ϵ and the isotropic one, $\epsilon_{\rm iso}$. This result indicates a need to reconsider any induction based only on the knowledge from the isotropic dissipation $\epsilon_{\rm iso}$. Experimentalists have to take into account this observation. They have to perform adequate experiments to describe more exactly the behavior of real turbulent flow. In this context, experiments on homogeneous axisymmetric turbulence are of great interest. Various investigations have confirmed that turbulent flows encountered in practice and that many laboratory flows are more closely approximated by axisymmetry than isotropy, for example, the fully developed turbulent channel flow, near the centerline [56]. Consequently, a more confident analysis can be actually achieved if it is based on the axisymmetric energy dissipation rate $\epsilon_{\text{axi},\rho=0}$ when r is parallel to λ, equation (32). All the second-order derivatives in equation (32) are measurable quantities.

ESS and scaling laws

As we have seen, Kolmogorov's equation is an exact relationship between the second and third order functions. When molecular viscosity becomes vanishingly small ($\nu \to 0$) and turbulent dissipation remains finite ($\epsilon\rangle 0$), the third order moment is proportional to the space separation r, equation (2). This equation is used to study experimentally and numerically the scaling exponents according to the ESS. In ESS, the n-order longitudinal velocity structure function $\langle \delta u_1^n \rangle$ is plotted against the third order structure function. The idea of plotting a structure function against the other was first put forward by Benzi *et al.* [19]. These authors succeeded in obtaining some very important results. Indeed, plotting a structure function against the other generally leads to a somewhat more extensive scaling range and allows a more confident determination of the exponents. Recently, Benzi *et al.* [57] have presented a generalized version of ESS (called G-ESS) which turns out to be much more universal and allowed the authors to draw a concrete theoretical framework of the energy cascade down to the smallest resolvable case (*i.e.* in a region where no anomalous scaling was supposed to be detected). Subsequent extensive investigations have confirmed that ESS holds at high as well as at low Reynolds number and it is characterized by the same scaling exponents of the velocity increments of fully developed turbulence [58]. Numerical and experimental studies supporting the generalized version of ESS have also been reported [59–61].

More recently, the scaling behaviour of the structure functions has been investigated numerically in the presence of anisotropic homogeneous turbulence [62]. Theoretical prediction checked on a number of laboratory experiments and direct numerical simulations has been reported [63]. It is shown that even in cases where ESS is not observed, a generalized self-scaling must be observed. Amati *et al.* [64] have argued that the failure to observe ESS in boundary layer turbulence resulted from the lack of isotropy of the flow. Their anisotropic datasets deviate from ESS. In fact, many practical flow do not comply with ESS. Under such circumstances, we came to the conclusion that with a more elaborate technique, it may be possible to test applicability of the ESS to non-isotropic homogeneous turbulence. Namely, the use of axisymmetric form of Kolmogorov's equation equation (39), which is an exact relation between the third-order structure functions $(T_1 + 2T_2)$ and the scale r. It allows us to study the scaling exponents of the n-order structure functions. In this way, one can plot the n-order structure functions against the sum of the third order scalar functions $T_1 + 2T_2$ instead of the third order longitudinal isotropic scalar function $\langle (\delta u_1)^3 \rangle$. In this case, it would be very interesting to examine the ESS. One can calculate the anomalous exponents associated with the n-order structure functions and discuss the self scaling observed in homogeneous axisymmetric turbulence.

8 Conclusion

Kinematics and dynamics of locally homogeneous axisymmetric turbulence have been derived with the assumption that the properties of the turbulence are invariant with respect to rotation about a preferred direction λ. In particular, the axisymmetric form of Kolmogorov's equation in the inertial range, equation (39). When the more constraining assumption of locally isotropy is made, equation (39) is reduced to Kolmogorov's equation. The corresponding axisymmetric limiting form is reduced to the isotropic limiting form of Kolmogorov's equation. We have summarized all the relations derived in Table 1.

This new analysis offers some means to investigators interested in the fundamental questions of turbulence. The results of the preceding sections suggest a number of tests (numerical or experimental verifications) which can determine whether the constraints of locally axisymmetric turbulence are satisfied or not. Remember that local axisymmetric turbulence is an assumption intermediate between local homogeneity for which relevant quantities are not measurable and local isotropy which is the simplest case of turbulence or a mathematical idealization. Consequently, only local axisymmetric turbulence makes it possible to test hypotheses on general effects of anisotropy in experiments. The present work then provides to the experimenters the theoretical bases which will be necessary for them to carry out these tests. The development of such a theory of axisymmetric turbulence may also be useful in establishing the circumstances under which isotropy may be expected to prevail. Moreover, there is a considerable suspicion that local isotropy is not a sinequanon condition for a correct description of many turbulent flows. In this context, it would be very interesting to test experimentaly as well as numericaly these new axisymmetric results and show their validity. These theoretical results are more general than the previous isotropic ones and they are important to better understand turbulent flows encountered in practice. They may have important consequences on various ideas on the fine structure of turbulence, such as ESS and the scaling laws for the n-order structure functions. They could argue and explain the failure to observe ESS in many laboratory experiments and DNS.

Table 1. Summary of the relationships.

locally homogeneous field and	anisotropy	locally axisymmetry	locally axisymmetry $\rho = 0$	locally isotropy
D_{ij}		equation (9)	$\lambda_i\lambda_j D_1 + D_2(\delta_{ij} - \lambda_i\lambda_j)$	$(D_{LL} - D_{NN})\frac{r_i r_j}{r^2} + D_{NN}\delta_{ij}$
D_{ijk}		equation (16)	$\lambda_i\lambda_j\lambda_k(T_1 - 3T_2) + T_2(\lambda_j\delta_{ik} + \lambda_i\delta_{jk} + \lambda_k\delta_{ij})$	$(D_{LLL} - 3D_{LNN})\frac{r_i r_j r_k}{r^3} + D_{LNN}\left(\frac{r_i}{r}\delta_{jk} + \frac{r_j}{r}\delta_{ik} + \frac{r_k}{r}\delta_{ij}\right)$
ϵ	$\epsilon = \nu\langle(\nabla_\alpha u_i)^2\rangle$ or equation (27 b)	equation (30)	$-\nu\left\langle\left(\frac{\partial u_1}{\partial x_1}\right)^2\right\rangle + 8\nu\left\langle\left(\frac{\partial u_2}{\partial x_2}\right)^2\right\rangle + 2\nu\left\langle\left(\frac{\partial u_1}{\partial x_2}\right)^2\right\rangle + 2\nu\left\langle\left(\frac{\partial u_2}{\partial x_1}\right)^2\right\rangle$	$15\nu\left\langle\left(\frac{\partial u_1}{\partial x_1}\right)^2\right\rangle$
Monin's equation	$\frac{\partial}{\partial r_\alpha}D_{ii\alpha} = 2\nu\frac{\partial^2}{\partial r_\alpha^2}D_{ii} - 4\epsilon$	equation (33)	$\frac{\partial}{\partial z}(T_1 + 2T_2) + \left(\frac{1}{\rho} + \frac{\partial}{\partial \rho}\right)(T_4 + T_5 + T_6) = -4\epsilon_{axi}$	$\left(\frac{2}{r} + \frac{\partial}{\partial r}\right)\left(4 + r\frac{\partial}{\partial r}\right)\left[\frac{1}{3}D_{LLL} - 2\nu\frac{\partial}{\partial r}D_{LL} + \frac{4}{15}\epsilon_{\rm iso} r\right] = 0$
Kolmogorov's equation in inertial range			$T_1 + 2T_2 = -\frac{4}{3}\epsilon_{axi} z$	$D_{LLL} = -\frac{4}{5}\epsilon_{\rm iso} r$
Limiting form $(r \to 0)$			$3\sum_{i=1}^{3}\alpha_i + 2\sum_{i=1}^{3}\zeta_i = 8\nu\sum_{i=1}^{3}e_i + 24\nu\sum_{i=1}^{3}d_i$	$\left\langle\left(\frac{\partial u_1}{\partial x_1}\right)^3\right\rangle = -2\nu\left\langle\left(\frac{\partial^2 u_1}{\partial x_1^2}\right)^2\right\rangle$
Relations between scalar functions of D_{ij}			D_1 and D_2 independent	$D_{NN}(r) = \left(1 + \frac{r}{2}\frac{\partial}{\partial r}\right)D_{LL}(r)$
Relations between scalar functions of D_{ijk}			T_1 and T_2 independent	$D_{LNN} = \frac{1}{6}\left(1 + r\frac{\partial}{\partial r}\right)D_{LLL}$

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